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Connections from trivializations

Dedicated to Professor Ivan Kolář on the occasion of his 80th birthday with respect and gratitude

Abstract. Let P be a principal fiber bundle with the basis M and with the structural group G. A trivialization of P is a section of P. It is proved that there exists only one gauge natural operator transforming trivializations of P into principal connections in P. All gauge natural operators transforming trivializations of P and torsion free classical linear connections on M into classical linear connections on P are completely described.

Introduction. All manifolds considered in the paper are assumed to be finite dimensional, Hausdorff, second countable, without boundary and smooth (of class C^∞). Maps between manifolds are assumed to be smooth (of class C^∞).

Let M be a manifold and let $p : P \to M$ (or shortly P) be a principal fibre bundle with the basis M and with the structure group G. Let $R : P\times G \to P$ be the right action.

A trivialization of P is a section $\sigma : M \to P$ of P.

A principal connection in P is a right invariant sub-bundle Γ of the tangent bundle TP of P such that $TP = VP \oplus_P \Gamma$, where $VP = \bigcup_{x\in M} TP_x \subset TP$ is the vertical bundle (over P) of $P \to M$, see [4]. The right invariance of Γ means that $TR_\xi(\Gamma) = \Gamma$ for any $\xi \in G$.

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Equivalently, a principal connection in P is a right invariant section $\Gamma : P \to J^1P$ of the first jet prolongation $\pi^1_0 : J^1P \to P$ of $P \to M$. Then the equivalence is given by the equality $\Gamma_p = imT_x\sigma$, where $\Gamma(p) = j^1_x\sigma$, $p \in P_x$, $x \in M$.

The right action of G on P induces a right action of G on the first jet prolongation J^1P of P by $v \cdot g = j^1_x(\sigma \cdot g)$, $v = j^1_x\sigma \in J^1P$, $g \in G$. The orbit of $j^1_x\sigma$ with respect to the action will be denoted by $[j^1_x\sigma]_G$. The fiber bundle $QP := J^1P/G = \{[j^1_x\sigma]_G \mid j^1_x\sigma \in J^1P\}$ over M of orbits of the right action of G on J^1P is called the principal connection bundle of P. Principal connections $\Gamma : P \to J^1P$ in P are in bijection with sections $\Gamma : M \to QP$ of $QP \to M$. The bijection is given by $\Gamma(x) := [j^1_x\sigma]_G$, where $\Gamma(p) = j^1_x\sigma$, $p \in P_x$, $x \in M$.

If $P = LM$ is the principal bundle (with the structure group $G = GL(m)$) of linear frames of a manifold M, a principal connection Λ in LM is called a classical linear connection on M.

Equivalently, a classical linear connection on M is a bilinear map $\nabla = \nabla^\Lambda : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that $\nabla_{fX}Y = f\nabla_XY$ and $\nabla_XfY = f\nabla_XY + X(f)Y$ for any vector fields $X, Y \in \mathcal{X}(M)$ on M and any map $f : M \to \mathbf{R}$, see [4].

A classical linear connection Λ on M is torsion-free if its torsion tensor T_Λ vanishes. (The torsion tensor T_Λ is a tensor field of type $(1, 2)$ on M given by $T_\Lambda(X, Y) = \nabla^\Lambda_XY - \nabla^\Lambda_YX - [X, Y]$.)

Equivalently, a classical linear connection on M is a linear section $\Lambda : TM \to J^1TM$ of the first jet prolongation $J^1TM \to TM$ of the tangent bundle TM of M, see [6].

In Section 1 of the present paper, we study the problem how a trivialization σ of P can induce a principal connection $A(\sigma)$ in P. This problem is reflected in the concept of gauge natural operators A in the sense of [6] producing principal connections $A(\sigma) : M \to QP$ in $P \to M$ from trivializations σ of P. We prove that any gauge natural operator A in question is given by $A(\sigma)(x) := [j^1_x\sigma]_G$.

In Section 2 of the present paper, we study the problem how a pair (σ, Λ) of a trivialization σ of P and a torsion free classical linear connection Λ on M can induce a classical linear connection $A(\sigma, \Lambda)$ on P. This problem is reflected in the concept of gauge natural operators A in the sense of [6] producing classical linear connections $A(\sigma, \Lambda)$ on P from trivializations σ of P by means of classical linear connections Λ on M. We describe completely all gauge natural operators A in question.

Natural operators producing connections have been studied in many papers, e.g. [1], [2], [3], [5], [6], etc.

1. Principal connections in P from trivializations of P. Let G be a Lie group and m a positive integer. Let $\mathcal{PB}_m(G)$ be the category of principal bundles with m-dimensional bases and with the structure group G and all (local) principal bundle isomorphisms with id_G as the group isomorphism. Let $\mathcal{FM}$ be the category of fibred manifolds and their fibred maps.

Any $\mathcal{PB}_m(G)$ object P over M induces the principal connection bundle $QP = J^1P/G$ over M (see Introduction) and any $\mathcal{PB}_m(G)$-morphism $f : P \to P^1$ with the base map $\underline{f} : M \to M^1$ induces fibred map $Qf : QP \to QP^1$ covering $\underline{f}$ defined by $Qf([j^1_x\sigma]_G) := [j^1_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})]_G$, $[j^1_x\sigma]_G \in QP$. The correspondence $Q : \mathcal{PB}_m(G) \to \mathcal{FM}$ is a gauge bundle functor in the sense of [6].

The general concept of gauge natural operators can be found in [6]. In particular, a gauge natural operator $A : id_{\mathcal{PB}_m(G)} \rightsquigarrow Q$ transforming trivializations of P into principal connections in P is a $\mathcal{PB}_m(G)$-invariant system of operators

$$A : C^\infty_M(P) \to C^\infty_M(QP)$$

for all $\mathcal{PB}_m(G)$-objects $P \to M$, where $C^\infty_M(P)$ is the set of all trivializations of P (possible $C^\infty_M(P) = \emptyset$ for some P) and $C^\infty_M(QP)$ is the set of all principal connections in P. The invariance of A means that if $\sigma \in C^\infty_M(P)$ and $\sigma^1 \in C^\infty_{M^1}(P^1)$ are f-related by an $\mathcal{PB}_m(G)$-map $f : P \to P^1$ with the base map $\underline{f} : M \to M^1$ (i.e. $f \circ \sigma = \sigma^1 \circ \underline{f}$), then $A(\sigma)$ and $A(\sigma^1)$ are Qf-related (i.e. $Qf \circ A(\sigma) = A(\sigma^1) \circ \underline{f}$). By [6], any (gauge) natural operator A is local and it can be extended uniquely on locally defined trivializations.

Example 1. For any $\mathcal{PB}_m(G)$ object P over M we have a function

$$D : C^\infty_M(P) \to C^\infty_M(QP),\ D(\sigma)(x) = [j^1_x\sigma]_G,\ \sigma \in C^\infty_M(P),\ x \in M.$$

The family $D : id_{\mathcal{PB}_m(G)} \rightsquigarrow Q$ of functions D for $\mathcal{PB}_m(G)$-objects P over M is a gauge natural operator (in question).

We have the following theorem.

Theorem 1. *The gauge natural operator $D : id_{\mathcal{PB}_m(G)} \rightsquigarrow Q$ (of Example 1) is the unique one, transforming trivializations of P into principal connections in P.*

Proof. Suppose that $A : id_{\mathcal{PB}_m(G)} \rightsquigarrow Q$ is a gauge natural operator. We have to show that $A(\sigma)(x) = [j^1_0(\sigma)]_G$ for any $\mathcal{PB}_m(G)$-object P over M, any $\sigma \in C^\infty_M(P)$ and any $x \in M$.

Because of the invariance of A with respect to the principal bundle charts, we may assume that $P = \mathbf{R}^m \times G$ (the trivial principal bundle over $M = \mathbf{R}^m$), $x = 0 \in \mathbf{R}^m$ and $\sigma(y) = (y, h(y))$, $y \in \mathbf{R}^m$, $h : \mathbf{R}^m \to G$. Then by the invariance of A with respect to the $\mathcal{PB}_m(G)$-morphism $f : \mathbf{R}^m \times G \to \mathbf{R}^m \times G$, $f(y, \xi) = (y, h(y)^{-1} \cdot \xi)$, we may assume that $\sigma(y) = (y, e_G)$, $y \in \mathbf{R}^m$.

Denote $A(\sigma)(0) = [j^1_0\rho]_G$, $\rho(0) = (0, e_G)$. Using the invariance of A with respect to $\mathcal{PB}_m(G)$-maps $a_t : \mathbf{R}^m \times G \to \mathbf{R}^m \times G$, $a_t(y, \xi) = (\frac{1}{t}y, \xi)$, $t > 0$, we get the homogeneous condition $A(\sigma)(0)) = [j^1_0(a_t \circ \rho \circ (\underline{a}_t)^{-1})]_G$,

$t > 0$. Putting $t \to 0$, we get $A(\sigma)(0) = [j^1_0(\sigma)]_G$. (More precisely, writing $\rho(y) = (y, k(y))$ with $k(0) = e_G$, we have $a_t \circ \rho \circ \underline{a}_t^{-1}(y) = (y, k(ty))$, and then $[j^1_0(a_t \circ \rho \circ \underline{a}_t^{-1})]_G \to [j^1_0(y, e_G)]_G = [j^1_0\sigma]_G$ if $t \to 0$.)

Theorem 1 is complete. □

2. Classical linear connections on P from trivializations of $P \to M$ by means of classical linear connections on M. Classical linear connections on a manifold M are principal connections in the principal bundle LM of linear frames on M. Thus classical linear connections on M are elements from $C^\infty_M(Q(LM))$. We denote the set of torsion free classical linear connections on M by $C^\infty_M(Q_\tau(LM))$.

By [6], a gauge natural operator $A : id_{\mathcal{PB}_m(G)} \times Q_\tau L\mathcal{B} \rightsquigarrow QL$ transforming pairs consisting of trivializations of P and torsion free classical linear connections on M into classical linear connections on P is a $\mathcal{PB}_m(G)$-invariant family of regular operators

$$A : C^\infty_M(P) \times C^\infty_M(Q_\tau(LM)) \to C^\infty_P(Q(LP))$$

for $\mathcal{PB}_m(G)$ objects P over M. The regularity of A means that A transforms smoothly parametrized families of pairs of trivializations of P and torsion free classical linear connections on M into smoothly parametrized families of classical linear connections on P. By [6], any (gauge) natural operator A is local and it can be extended uniquely on locally defined pairs (σ, Λ) in question.

Example 2. Let P be an $\mathcal{PB}_m(G)$-object over M. In Sect. 54.7 in [6], the authors construct canonically the classical linear connection $N(D, \Lambda)$ on P from a principal connection D in P by means of a classical linear connection Λ on M. So, using a trivialization $\sigma \in C^\infty_M(P)$ of P and a torsion free classical linear connection Λ on M we can produce a classical linear connection

$$Q(\sigma, \Lambda) := N(D(\sigma), \Lambda)$$

on P, where $D(\sigma)$ is the principal connection in P from σ as in Example 1. The family $Q : id_{\mathcal{PB}_m(G)} \times Q_\tau L\mathcal{B} \rightsquigarrow QL$ of functions Q is a gauge natural operator (in question).

Example 3. Let

$$\Delta : G \to T_{(0,e_G)}(\mathbf{R}^m \times G) \otimes T^*_{(0,e_G)}(\mathbf{R}^m \times G) \otimes T^*_{(0,e_G)}(\mathbf{R}^m \times G)$$

be a smooth map such that $\Delta(\xi)$ is a $GL(\mathbf{R}^m) \times \{id_G\}$-invariant tensor of type $(1, 2)$ on $\mathbf{R}^m \times G$ at $(0, e_G)$ for any $\xi \in G$. Then we have gauge natural operator

$$A^{<\Delta>} : id_{\mathcal{PB}_m(G)} \times Q_\tau L\mathcal{B} \rightsquigarrow QL$$

defined as follows.

Let $\sigma \in C^\infty_M(P)$, $\Lambda \in C^\infty_M(Q_\tau(LM))$, $p \in P_x$, $x \in M$. There is a principal bundle chart $\varphi : P_{|U} \to \mathbf{R}^m \times G$ with $\varphi(p) = (0, e_G)$ and sending $\sigma_{|U}$ into a constant section $\sigma^o = (id_{\mathbf{R}^m}, \xi^o) \in C^\infty_{\mathbf{R}^m}(\mathbf{R}^m \times G)$ for some $\xi^o \in G$. Clearly, ξ^o is defined by $\sigma(x) = R_{\xi^o}(p)$. Denote the base map of φ by $\underline{\varphi} : U \to \mathbf{R}^m$. Let Λ' be the image of $\Lambda_{|U}$ by $\underline{\varphi}$ and let ψ be a Λ'-normal coordinate system with center 0. Replacing φ by $(\psi \times id_G) \circ \varphi$, we may additionally assume that $\underline{\varphi}$ is a normal coordinate system of Λ with center x. Recalling that QLP is the affine bundle with $TP \otimes T^*P \otimes T^*P$ as the corresponding vector bundle, we put

$$A^{<\Delta>}(\sigma, \Lambda)(p) := Q(\sigma, \Lambda)(p) + T_{(0,e_G)}\varphi^{-1} \otimes T^*_{(0,e_G)}\varphi^{-1} \otimes T^*_{(0,e_G)}\varphi^{-1}(\Delta(\xi^o)) ,$$

where Q is as in Example 2. If φ_1 is another such chart, then $\varphi_1 = (B \times id_G) \circ \varphi$ for a linear isomorphism $B \in GL(\mathbf{R}^m)$. So, the definition of $A^{<\Delta>}(\sigma, \Lambda)(p)$ is independent of the choice of φ because of the invariance of $\Delta(\xi^o)$.

We have the following theorem.

Theorem 2. *Let $A : id_{\mathcal{PB}_m} \times Q_\tau L\mathcal{B} \rightsquigarrow QL$ be a gauge natural operator. There is the smooth map $\Delta : G \to T_{(0,e_G)}(\mathbf{R}^m \times G) \otimes T^*_{(0,e_G)}(\mathbf{R}^m \times G) \otimes T^*_{(0,e_G)}(\mathbf{R}^m \times G)$ such that $\Delta(\xi)$ is $GL(\mathbf{R}^m) \times \{id_G\}$-invariant for any $\xi \in G$ and $A = A^{<\Delta>}$.*

The maps Δ (in question) are in bijection with the triples (a, b, c) of smooth maps $a, b : G \to Lie(G)^*$ and $c : G \to Lie(G) \otimes Lie(G)^* \otimes Lie(G)^*$, where $Lie(G)$ is the Lie algebra of G. So, if we choose the basis in $Lie(G)$, the gauge natural operators A (in question) are in bijection with the $(2k + k^3)$-tuples of smooth maps $G \to \mathbf{R}$, where $k = \dim(G)$.

Proof. We have to put

$$\Delta(\xi^o) := A(\sigma^o, \Lambda^o)(0, e_G) - Q(\sigma^o, \Lambda^o)(0, e_G) ,$$

where $\xi^o \in G$, $\sigma^o = (id_{\mathbf{R}^m}, \xi^o)$ and Λ^o is the torsion free flat classical linear connection on $\mathbf{R}^m$ and Q is as in Example 2. Then Δ is smooth in ξ^o (as A is regular) and $\Delta(\xi^o)$ is $GL(\mathbf{R}^m) \times \{id_G\}$-invariant because A, Q, σ^o, Λ^o, 0 and e_G are. We prove that $A = A^{<\Delta>}$.

It is sufficient to show that $A(\sigma, \Lambda)(p) = A^{<\Delta>}(\sigma, \Lambda)(p)$ for any $\mathcal{PB}_m(G)$-object P over M, $\sigma \in C^\infty_M(P)$, $\Lambda \in C^\infty_M(Q_\tau(LM))$, $p \in P_x$, $x \in M$. Because of the invariance of A and $A^{<\Delta>}$ with respect to chart φ as in Example 3, we may assume that $P = \mathbf{R}^m \times G$, $M = \mathbf{R}^m$, $\sigma = \sigma^o = (id_{\mathbf{R}^m}, \xi^o)$, Λ is a torsion free classical linear connection on $\mathbf{R}^m$ with $\Lambda(0) = \Lambda^o(0)$, $p = (0, e_G)$, $x = 0$.

The invariance of A with respect to the $\mathcal{PB}_m(G)$-maps a_t from the proof of Theorem 1 gives the homogeneous condition

$$A(\sigma^o, (a_t)_*\Lambda)(0, e_G) = Ta_t \otimes T^*a_t \otimes T^*a_t(A(\sigma^o, \Lambda)(0, e_G))$$

for $t > 0$. Because of the non-linear Petree theorem (see Corollary 19.8 in [6]) we may assume that the Cristoffel symbols Λ are polynomial maps. Then by the homogeneous function theorem (see [6]) we deduce that $A(\sigma^o,-)(0,e_G)$ depends on $\Lambda(0)$ (and similarly for $A^{<\Delta>}$ instead of A). So,

$$\begin{aligned} A(\sigma^o,\Lambda)(0,e_G) &= A(\sigma^o,\Lambda^o)(0,e_G) = A^{<\Delta>}(\sigma^o,\Lambda^o)(0,e_G) \\ &= A^{<\Delta>}(\sigma^o,\Lambda)(0,e_G). \end{aligned}$$

We else describe all maps Δ from Example 3.

Let Δ be a map in question. We see that $T_{(0,e_G)}(\mathbf{R}^m\times G)=\mathbf{R}^m\oplus \mathit{Lie}(G)$ modulo the standard identification. Then for any $\xi\in G$, $\Delta(\xi)$ can be considered as the $GL(\mathbf{R}^m)\times\{id_{\mathit{Lie}(G)}\}$ invariant tensor $\Delta(\xi)$ from $(\mathbf{R}^m\oplus \mathit{Lie}(G))\otimes(\mathbf{R}^m\oplus \mathit{Lie}(G))^*\otimes(\mathbf{R}^m\oplus \mathit{Lie}(G))^* = (\mathbf{R}^m\otimes\mathbf{R}^{m*}\otimes\mathbf{R}^{m*})\oplus(\mathbf{R}^m\otimes\mathbf{R}^{m*}\otimes \mathit{Lie}(G)^*)\oplus(\mathbf{R}^m\otimes \mathit{Lie}(G)^*\otimes\mathbf{R}^{m*})\oplus(\mathbf{R}^m\otimes \mathit{Lie}(G)^*\otimes \mathit{Lie}(G)^*))\oplus(\mathit{Lie}(G)\otimes\mathbf{R}^{m*}\otimes\mathbf{R}^{m*})\oplus(\mathit{Lie}(G)\otimes\mathbf{R}^{m*}\otimes \mathit{Lie}(G)^*)\oplus(\mathit{Lie}(G)\otimes \mathit{Lie}(G)^*\otimes\mathbf{R}^{m*})\oplus(\mathit{Lie}(G)\otimes \mathit{Lie}(G)^*\otimes \mathit{Lie}(G)^*)$.

Thus $\Delta(\xi)=(\Delta_1(\xi),\dots,\Delta_8(\xi))$, where $\Delta_i(\xi)$ for $i=1,\dots,8$ are the respective components of $\Delta(\xi)$ with respect to the above decomposition. By the $GL(\mathbf{R}^m)\times\{id_{\mathit{Lie}(G)}\}$-invariance, $\Delta_2(\xi)$, $\Delta_3(\xi)$ and $\Delta_8(\xi)$ may be not zero, only. Moreover, $\Delta_8(\xi)$ may be arbitrary (smoothly depending on ξ), $\Delta_2(\xi)=id_{\mathbf{R}^m}\otimes\delta_2(\xi)$ and $\Delta_3(\xi)=\delta_3(\xi)\otimes id_{\mathbf{R}^m}$ (modulo the permutation), where $\delta_2(\xi)$ and $\delta_3(\xi)$ are arbitrary elements from $\mathit{Lie}(G)^*$ (smooth in ξ). Then the maps Δ from Example 3 are in bijection with the triples (a,b,c) of smooth maps $a,b:G\to \mathit{Lie}(G)^*$ and $c:G\to \mathit{Lie}(G)\otimes \mathit{Lie}(G)^*\otimes \mathit{Lie}(G)^*$, $a=\delta_2$, $b=\delta_3$, $c=\Delta_8$. □

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